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Progressive internal waves on slopes

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The refraction of progressive internal waves on sloping bottoms is treated for the case of constant Brunt–Väisälä frequency. In two dimensions simple, explicit expressions for the changing wavelengths and amplitudes are found. For small slopes, the solutions reduce to simple propagating waves at infinity.

The singularity along a characteristic is shown to be removable, though the solutions are now inhomogeneous waves. The viscous boundary layers of the wedge geometry are briefly considered with the inviscid solutions remaining as interior solutions.

A theory valid for small slopes is obtained for three-dimensional waves. The waves are refracted in the usual manner, turning parallel to the beach in shallow water.

1. Introduction

In a recent note (Wunsch 1968, which we will call (I)), it was shown that in an inviscid Boussinesq fluid of constant Brunt–Väisälä frequency, the expressions for the propagation of two-dimensional internal waves up a sloping 'beach' took on a particularly simple form. Attention in (I) was focused on the standing modes of a wedge with rigid upper and lower boundaries.

In subsequent attempts to reproduce the results experimentally a propagating mode solution appeared to be more appropriate and more in accord with the few oceanic observations available. Furthermore, the line of high shear predicted by the standing wave solutions was never reproduced in the laboratory. The purpose of this discussion is to examine the propagating solutions of a wedge geometry. In particular we show that the characteristic singularity is removable with an inviscid theory. The possible viscous corrections to the solutions will be briefly examined. Finally, the theory is extended to the problem of the refraction of three-dimensional internal waves on slopes, valid in the limit of small slopes.

2. Progressive waves

In a Boussinesq fluid of mean density $\rho_0(z)$, the perturbation equations take the form:

$$u_t = p_x/\rho_0 + \nu\nabla^2 u, \tag{1}$$

$$v_t = -p_y/\rho_0 + \nu\nabla^2 v, \tag{2}$$

$$w_t = -p_z/\rho_0 - g\rho'/\rho_0 + \nu\nabla^2 w, \tag{3}$$

$$u_x + v_y + w_z = 0, \tag{4}$$

$$\rho'_t + w\rho_{0z} = 0. \tag{5}$$

ρ' is the perturbation density. We assume a time-dependence $e^{-i\sigma t}$.

It was shown in (I) that within rigid boundaries defined by $z = 0$, $z = -\gamma x$, $x > 0$, exact two-dimensional inviscid solutions existed in the form of standing waves. In terms of a stream function ψ,

$$\psi_{zz} - c^{-2}\psi_{xx} = 0 \tag{6}$$

and

$$\left.\begin{aligned} \psi^A &= \sin\frac{2n\pi}{\ln\Delta}\ln(cx-z) - \sin\frac{2n\pi}{\ln\Delta}\ln(cx+z), \\ \psi^B &= -\cos\frac{2n\pi}{\ln\Delta}\ln(cx-z) + \cos\frac{2n\pi}{\ln\Delta}\ln(cx+z), \end{aligned}\right\} \tag{7}$$

where

$$c^2 = \frac{\sigma^2}{N^2-\sigma^2}, \quad N^2 = -\frac{g}{\rho_0}\rho_{0z} = \text{constant}, \quad n = \text{integer}, \quad \Delta = \frac{c+\gamma}{c-\gamma}.$$

If the fluid is rotating, and the wave propagation remains two-dimensional, c^2 becomes simply $(\sigma^2 - f^2)/(N^2-\sigma^2)$, the solutions being otherwise unchanged.

These are not the only standing waves but appear to be the simplest and most easily interpretable. They are singular along the line $z = -cx$ (corresponding to a 'critical angle' $\alpha = \tan^{-1} 1/c$). Perhaps it should be pointed out that (7) are also solutions exterior to a wedge, with singularities along the lines $z = -cx$ and $z = +cx$.

If we restrict our attention for the moment to wedge angles less than the critical angle (we will call this a 'subcritical wedge'), then we can make up two propagating solutions out of the standing solutions. We have

$$\psi^+ = i\psi^A - \psi^B = A_+[\exp\{iq\ln(cx-z)\} - \exp\{iq\ln(cx+z)\}], \tag{8}$$

$$\psi^- = i\psi^A + \psi^B = A_-[\exp\{-iq\ln(cx-z)\} - \exp\{-iq\ln(cx+z)\}], \tag{9}$$

where A_+ and A_- are constant, $q = 2n\pi/\ln\Delta$.

The phase velocities (and as we shall show, the energy flux) of these two solutions are down-slope and up-slope respectively. That they satisfy the boundary conditions $\psi = 0$ on the walls, can be seen from

$$\psi^+ = 2iA_+\sin\left[\tfrac{1}{2}q\ln\left(\frac{cx-z}{cx+z}\right)\right]\exp\{i\tfrac{1}{2}q\ln(c^2x^2-z^2)\}, \tag{10}$$

$$\psi^- = 2iA_-\sin\left[\tfrac{1}{2}q\ln\left(\frac{cx-z}{cx+z}\right)\right]\exp\{-i\tfrac{1}{2}q\ln(c^2x^2-z^2)\}. \tag{11}$$

The singularity at $x = z = 0$ remains. The ψ^- solutions appear to be the most interesting physically, so we shall concentrate attention upon them. The changes for the ψ^+ solutions are obvious. One of the modes is shown in figure 1. The structure of these waves is similar to the standing modes, and reference is made to (I) for further illustrations. Related solutions have been given by Magaard (1962) and Sandstrom (1966).†

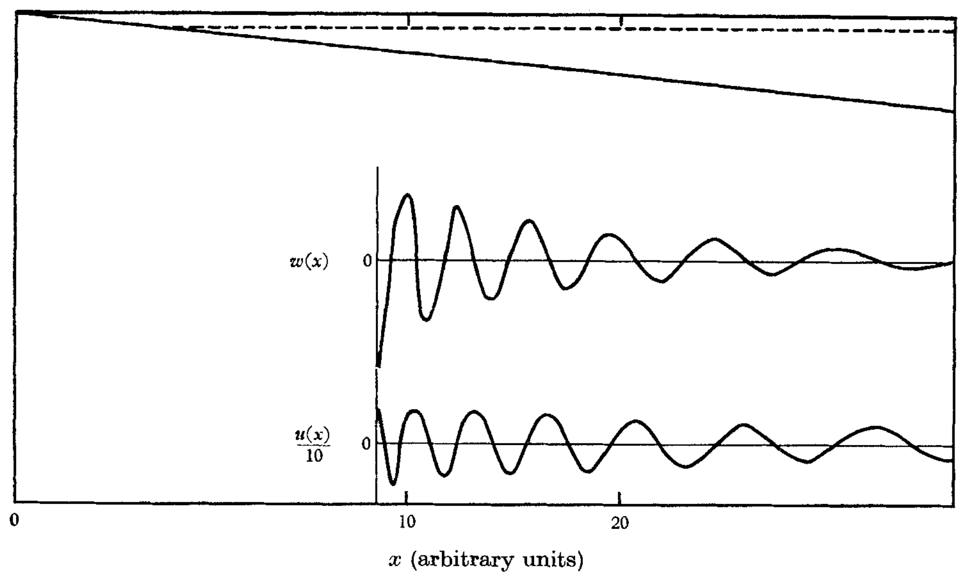

FIGURE 1. Mode 1 of the case $c = 1{\cdot}0$, $\gamma = 1/10$, showing horizontal and vertical particle velocities along the dashed line. The horizontal velocity is shown at 1/10 scale. Note the decrease in wavelength as the corner is approached, and the increase in amplitude. ($z = -0{\cdot}5$.)

The relationship between these wedge solutions and the classical solutions (Lamb 1932, p. 738) in a flat-bottomed fluid is not obvious. In fact, for small slopes, they are asymptotically the same, the wedge solutions reducing to simple propagating waves at infinity.

Consider a shallow wedge, expressed by the condition $\gamma/c \ll 1$. Then we have

$$\ln\left(\frac{c+\gamma}{c-\gamma}\right) = \ln\left(\frac{1+\gamma/c}{1-\gamma/c}\right) \sim \frac{2\gamma}{c}.$$

Far from the corner where $cx \gg z$, we have approximately,

$$\begin{aligned}\psi^- &\sim A_-\left(\exp\left\{-\frac{in\pi c}{\gamma}\left[\ln(cx)-\frac{z}{cx}\right]\right\}-\exp\left\{-\frac{in\pi c}{\gamma}\left[\ln(cx)+\frac{z}{cx}\right]\right\}\right)\\ &= 2iA_-\exp\left\{-\frac{in\pi c}{\gamma}\ln(cx)\right\}\sin\frac{n\pi}{\gamma}\frac{z}{x}.\end{aligned}$$

† After this work was completed, it was pointed out to me by Prof. Harvey Greenspan that he had solved an analogous problem for a vibrating string in Greenspan (1963).

The local x wave-number k_x is

$$k_x = d/dx\left(\frac{n\pi c}{\gamma}\ln(cx)\right) = \frac{n\pi c}{\gamma x}.$$

The depth of the fluid is $h = \gamma x$; locally we have

$$\psi^- \sim 2iA_-\exp\left\{-i\frac{n\pi c}{h}x\right\}\sin\frac{n\pi z}{h}, \tag{12}$$

which is the solution for a flat-bottom. Hence in the shallow wedge, it is easy to make the correspondence between a mode in a flat region, and the equivalent mode in a shelving region.

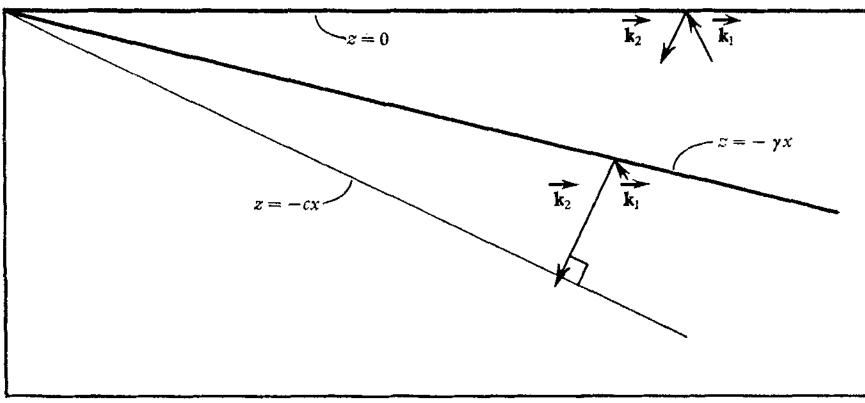

FIGURE 2. The wave-number change of the upward and downward propagating waves of a given mode. The downward wave-number goes to infinity at the critical angle.

Returning to the wedge itself, it is useful to examine the local wave-numbers. Solution (9) is made up of two propagating waves with local wave-numbers

$$\mathbf{k}_1 = \left(-\frac{qc}{cx-z},\ \frac{q}{cx-z}\right), \tag{13}$$

$$\mathbf{k}_2 = \left(-\frac{qc}{cx+z},\ -\frac{q}{cx+z}\right). \tag{14}$$

The x-component of both of these wave-numbers is negative, hence the phase velocity is up-slope. With z negative, the z component of $\mathbf{k}_1$ is positive, hence propagation is upward; the z component of $\mathbf{k}_2$ is negative, and the z-phase velocity is downward. At $z = 0$, $|\mathbf{k}_1| = |\mathbf{k}_2|$. At $z = -\gamma x$, $|\mathbf{k}_1| < |\mathbf{k}_2|$ and as $\gamma \to c$, $|\mathbf{k}_2| \to \infty$. The relationships are shown in figure 2. The relative magnitudes of $\mathbf{k}_1$ and $\mathbf{k}_2$ can be shown to satisfy the reflexion conditions discussed by Phillips (1966), for reflexion from horizontal and sloping surfaces.

3. Critical angle singularity

The singularity at $z = -cx$, can be traced to the growth of the $\mathbf{k}_2$ wave-number. The line $z = -cx$ is a branch point of $\ln\zeta$, $\zeta = cx+z$. If the wedge is supercritical, it is necessary to specify which sheet of the Riemann surface of the logarithm is to be chosen for $z+cx<0$. By the simple expedient of introducing an infinitesimally small amount of friction proportional to the velocity, we can make c complex. The independent variable ζ is then displaced off the real axis, and we find that we should choose

$$\ln(-|\zeta|) = \ln|\zeta| + i\pi$$

to give a physically consistent result.

Suppose that $\gamma > c$; q will now be complex and we write $q = \eta + i\delta$. The boundary condition on the slope requires

$$\sin\left\{\frac{\eta+i\delta}{2}[\ln|\Delta'|+i\pi]\right\} = 0, \quad \Delta' = \frac{c-\gamma}{c+\gamma}$$

or

$$\frac{\eta+i\delta}{2}(\ln|\Delta'|+i\pi) = n\pi \quad (n \equiv \text{integer}). \tag{15}$$

We find

$$\eta = \frac{2n\pi\ln|\Delta'|}{(\ln|\Delta'|)^2+\pi^2} \tag{16}$$

and

$$\delta = \frac{-2n\pi^2}{(\ln|\Delta'|)^2+\pi^2}. \tag{17}$$

Note that we have chosen to use the quantity $\Delta' = (c-\gamma)/(c+\gamma)$ rather than $\Delta = (c+\gamma)/(c-\gamma)$. The stream function is

$$\psi^- = A_-[(cx-z)^\delta \exp\{-i\eta\ln(cx-z)\} - (cx+z)^\delta \exp\{-i\eta\ln(cx+z)\}]. \tag{18}$$

If we can make δ a positive quantity, then the right-hand term of (18) will have a zero at $z+cx=0$, cancelling the singularity. If n is thus supposed to be a negative integer, δ is positive and ψ^- will be continuous. Note that since

$$|\Delta'| < 1, \quad \ln|\Delta'| < 0,$$

η is positive and the up-slope propagation sense is preserved. (The ψ^+ solutions demand the retention of $\ln|\Delta|$.)

The velocity field is

$$\begin{aligned} w = \psi^-_x = A_-(&-i(\eta+i\delta)(cx-z)^{\delta-1}c\exp\{-i\eta\ln(cx-z)\} \\ &+ i(\eta+i\delta)(cx+z)^{\delta-1}c\exp\{-i\eta\ln(cx+z)\}) \end{aligned} \tag{19}$$

and

$$\begin{aligned} u = -\psi^-_z = -A_-(&i(\eta+i\delta)(cx-z)^{\delta-1}\exp\{-i\eta\ln(cx-z)\} \\ &+ i(\eta+i\delta)(cx+z)^{\delta-1}\exp\{-i\eta\ln(cx+z)\}). \end{aligned} \tag{20}$$

These are everywhere bounded if $\delta > 1$, i.e.

$$\left.\begin{aligned} \frac{-2n\pi^2}{(\ln|\Delta'|)^2+\pi^2} - 1 > 0, \\ \text{or} \qquad -n > \frac{(\ln|\Delta'|)^2}{2\pi^2} + \tfrac{1}{2}. \end{aligned}\right\} \tag{21}$$

The velocities are continuous if (21) is satisfied. If, in addition, we require that their first derivatives also be continuous, we arrive at

$$-n > \frac{(\ln|\Delta'|)^2}{\pi^2} + 1.$$

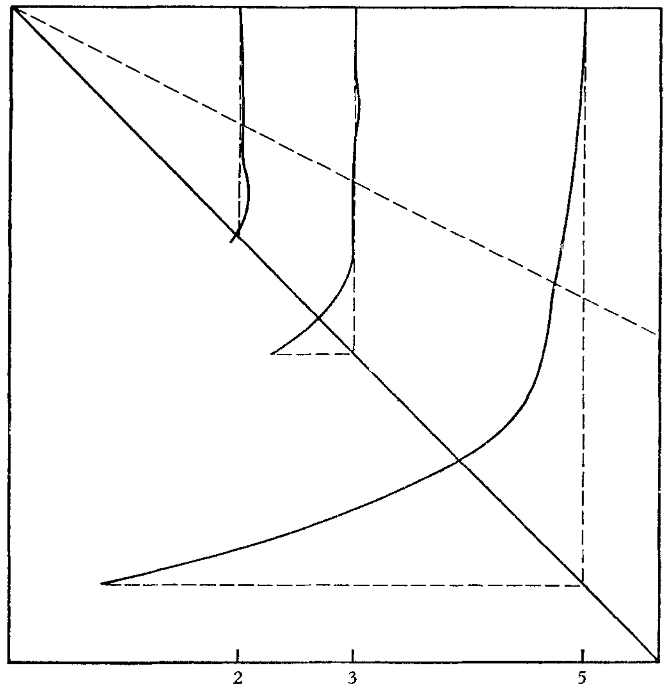

FIGURE 3. The in-phase vertical particle velocity w at selected places in the supercritical wedge $\gamma = 1{\cdot}0$, $c = 0{\cdot}5$, for $n = -2$. The shear near the slope is pronounced.

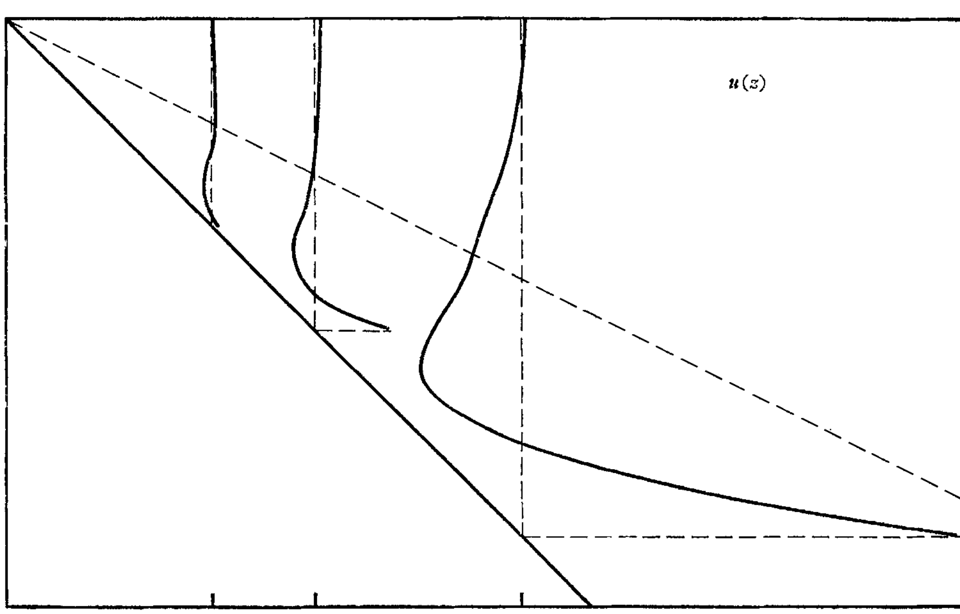

FIGURE 4. The in-phase horizontal particle velocity u for the case illustrated in figure 3.

This assures that the advective terms of the equations of motion do not introduce a fictitious force into the modes. The $(n+1)$st or higher derivative of the velocity vector will always be discontinuous no matter how large n is chosen. Formally, as $c \to \gamma$ from above, the minimum permissible value of n approaches minus infinity. If, in fact, discontinuous modes exist, then one expects that friction would effectively damp them out; hence this seems to be an unusual situation in which friction would require longevity in the high modes rather than the low. One expects that at frequencies near critical, higher modes would be present in a system in preference to the low. However, this is speculation and may be pushing a simple theory too far.

Unlike the subcritical solutions considered above and in (I), these supercritical modes are bounded in the corner; in fact, they are zero there. On the other hand, they grow algebraically as $x, z \to \infty$. The region exterior to the wedge is also everywhere nonsingular, except at infinity.

An example is shown in figures 3 and 4. The intensification on the bottom is very obvious. This can be traced back to the way the rays reflect from a slope.

The direction of energy flux in these waves is of interest. The x-component of energy flux is

$$I_x = \tfrac{1}{2}(pu^* + p^*u),$$

where * denotes conjugation. This is

$$\begin{aligned} I_x = -\frac{\sigma\rho_0}{2c}\bigg(&\frac{q+q^*}{cx-z}\exp\{-iq\ln(cx-z)+iq^*\ln^*(cx-z)\} \\ &+\frac{q+q^*}{cx+z}\exp\{-iq\ln(cx+z)+iq^*\ln^*(cx+z)\} \\ &+\frac{q^*(cx-z)+q(cx+z)}{c^2x^2-z^2}\exp\{-iq\ln(cx-z)+iq^*\ln^*(cx+z)\} \\ &+\frac{q^*(cx+z)+q(cx-z)}{c^2x^2-z^2}\exp\{iq^*\ln^*(cx-z)-iq\ln(cx+z)\}\bigg). \qquad (22) \end{aligned}$$

Note that $\ln(cx-z)$ is actually real.

This is shown numerically in figure 5 for $\gamma = 1$, $c = 0{\cdot}5$ and $n = -2$. The sign of the energy flux changes as the slope is approached, being up-slope far from the slope, and down-slope nearby. It is no longer possible to consider these as simple progressive waves, the polynomial coefficients rendering them rather complicated. Apparently the energy flux balances in such a way that all the energy is reflected before encountering the corner, rendering a singularity unnecessary. In the solutions for subcritical angles given above, the flux is all up-slope and the corner singularity is required.

It may be easily verified that in the limiting case of a right-angled corner, the solutions become

$$\psi^- = (cx-z)^{-2n} - (cx+z)^{-2n},$$

the wavelength having become infinite.

Examination of the ψ^+ solutions indicates that the sense of the energy flux is just the reverse of that for the ψ^- solutions.

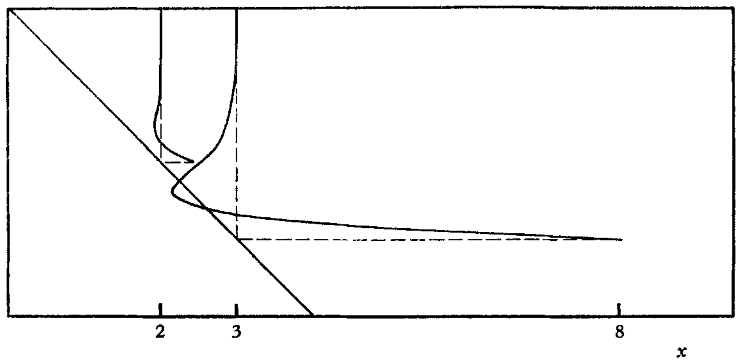

FIGURE 5. The horizontal energy flux for $\gamma = 1{\cdot}0$, $c = 0{\cdot}5$, $n = -2$.

4. Frictional boundary layers

The inviscid steady equation governing the stream function is hyperbolic in the space co-ordinates. As was pointed out in (I), this is very unusual and can lead to paradoxes. For this reason, it is interesting to examine the viscous equations, which are not hyperbolic. In particular, we look for a boundary-layer character, which will leave our inviscid results as valid interior solutions.

Equations (1)–(5) yield for non-zero ν, and the continued assumption $\partial/\partial y = 0$ an equation in the stream function that is now

$$\nu\nabla^4\psi_t - \nabla^2\psi_{tt} - N^2\psi_{xx} = 0. \tag{23}$$

We scale this with a length L, taken to be a local wedge depth, and a time

$$\left.\begin{aligned} &t \sim (1/N),\\ &(x,z) = L(x',z'), \quad t = t'/N\\ \text{and} \quad &R^{-1}\nabla^4\psi_{t'} - \nabla^2\psi_{t't'} - \psi_{x'x'} = 0, \quad R = NL^2/\nu,\end{aligned}\right\} \tag{24}$$

where R is a Reynolds number. Let us introduce a co-ordinate rotation

$$\left.\begin{aligned}\xi &= x'\cos\alpha - z'\sin\alpha,\\ \eta &= z'\cos\alpha + x'\sin\alpha,\end{aligned}\right\} \tag{25}$$

so that $\eta = 0$ is a radial line at angle α within the wedge. The line is taken to be a boundary. We have with $\psi \propto e^{-i\sigma t}$, $\sigma = O(1)$

$$-i\sigma R^{-1}\left\{\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right)^2\psi\right\} + \sigma^2\left(\frac{\partial^2}{\partial\eta^2}+\frac{\partial^2}{\partial\xi^2}\right)\psi$$
$$-\left(\cos^2\alpha\frac{\partial^2}{\partial\xi^2} + \sin^2\alpha\frac{\partial^2}{\partial\eta^2} + 2\sin\alpha\cos\alpha\frac{\partial^2}{\partial\eta\,\partial\xi}\right)\psi = 0. \tag{26}$$

Depending upon the choice of α, a number of boundary-layer balances are possible. $\alpha = 0$, corresponds to the top boundary. Here we have a boundary layer of thickness $O(R^{-\frac{1}{2}})$. This is an ordinary Stokes boundary layer, governed by

$$\tilde{\psi}^U_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}} + i\sigma\tilde{\psi}^U_{\tilde{\eta}\tilde{\eta}} = 0,$$

$\tilde{\eta}$ being a stretched co-ordinate with solution corresponding to the ψ^- waves of

$$\tilde{\psi}^U(x',z') = -\frac{iq}{cx'}\left(\frac{2}{\sigma}\right)^{\frac{1}{2}}\frac{R^{-\frac{1}{2}}}{1-i}A_-\exp\{-iq\ln(cx')+(2\sigma)^{\frac{1}{2}}(1-i)R^{\frac{1}{2}}z'\}. \quad (27)$$

There is an $O(R^{-\frac{1}{2}})$ flow into the boundary. Note the intensification as $x'\to 0$. If $(\sigma^2-\sin^2\alpha) > O(R^{-\frac{1}{2}})$, then a similar boundary layer is appropriate on the slope, yielding a lower boundary layer of form

$$\tilde{\psi}^L = \frac{iq(2\sigma)^{\frac{1}{2}}R^{-\frac{1}{2}}A_-}{(\sigma^2-\sin^2\alpha)^{\frac{1}{2}}(-1+i)\xi}\left\{\frac{c\sin\alpha-\cos\alpha}{c\cos\alpha+\sin\alpha}\exp\{-iq\ln(c\cos\alpha+\sin\alpha)\xi\}\right.$$
$$\left.-\left(\frac{c\sin\alpha+\cos\alpha}{c\cos\alpha-\sin\alpha}\right)\exp[-iq\ln(c\cos\alpha-\sin\alpha)\xi]\right\}$$
$$\times\exp\left\{(-1+i)\,|2|^{\frac{1}{2}}\left(\frac{\sigma^2-\sin^2\alpha}{\sigma}\right)^{\frac{1}{2}}R^{\frac{1}{2}}\eta\right\}. \quad (28)$$

When $(\sigma^2-\sin^2\alpha) < O(R^{-\frac{1}{2}})$, the slope is near critical; there is a balance in the equation over a distance $R^{-\frac{1}{3}}$, the boundary-layer dynamics being governed by

$$-i\sigma\tilde{\psi}^L_{\tilde{\eta}\tilde{\eta}\tilde{\eta}\tilde{\eta}}-2\tilde{\psi}^L_{\tilde{\eta}\xi} = 0. \quad (29)$$

The non-separability is reminiscent of a 'corner'. Here a simple analytic expression is not obvious.

As the apex of the wedge is approached, the scale length L begins to approach the boundary-layer thickness, and the boundary-layer analysis breaks down.

5. Three-dimensional refraction

We will consider the problem of waves at oblique incidence to the 'beach', limited to the inviscid case. With a three-dimensional dependence, a stream function is no longer available. The equation governing the vertical velocity w is

$$w_{zz}-(1/c^2)(w_{xx}+w_{yy}) = 0, \quad (30)$$

c as before. The boundary conditions are now

$$w = 0 \quad \text{at} \quad z = 0,$$

and $$\gamma u+w = 0 \quad \text{on} \quad z = -\gamma x \quad (x > 0). \quad (31)$$

Let $$w = \tilde{w}(x,z)\,e^{ily},$$

where $$\tilde{w}_{zz}-\frac{1}{c^2}\tilde{w}_{xx}+\frac{l^2}{c^2}\tilde{w} = 0, \quad (32)$$

and l is constant. This is a Klein–Gordon equation. To solve it, let

$$\xi = iz/c$$

and $$\tilde{w}_{\xi\xi}+\tilde{w}_{xx}-l^2\tilde{w} = 0. \quad (33)$$

If we now let $$r = (x^2+\xi^2)^{\frac{1}{2}} = \left(x^2-\frac{z^2}{c^2}\right)^{\frac{1}{2}}$$

and $$\phi = \tan^{-1}\frac{\xi}{x} = \tan^{-1}\frac{iz}{cx},$$

we have
$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \tilde{w}}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2\tilde{w}}{\partial\phi^2}-l^2\tilde{w}=0, \tag{34}$$

the ordinary Helmholtz equation, with solutions

$$\tilde{w}(r,\phi)=\begin{Bmatrix} I_\nu(lr) \\ I_{-\nu}(lr)\end{Bmatrix}\sin\nu\phi, \tag{35}$$

where we have chosen $\sin\nu\phi$ in anticipation of the upper boundary condition. I_ν is the modified Bessel function. Instead of $I_{-\nu}$, we could use as a second solution

$$K_\nu(J)=\frac{\pi}{2}\frac{I_{-\nu}(J)+I_\nu(J)}{\sin\nu\pi}.$$

Rewriting (35) in the original co-ordinate system, we have

$$w(x,y,z)=\begin{Bmatrix} I_\nu(l(x^2-z^2/c^2)^{\frac{1}{2}}) \\ I_{-\nu}(l(x^2-z^2/c^2)^{\frac{1}{2}})\end{Bmatrix}\sin\nu\left(\tan^{-1}\frac{iz}{cx}\right)e^{ily}. \tag{36}$$

Since $\tan\phi = iz/cx$, let $\phi = i\beta$, then

$$\tan i\beta = i\tanh\beta = iz/cx,$$

or
$$\beta=\tanh^{-1}z/cx$$

and
$$w=A_\pm I_{\pm\nu}[l(x^2-z^2/c^2)^{\frac{1}{2}}]\sin[i\nu(\tanh^{-1}z/cx)]\,e^{ily}.$$

Noting that
$$\tanh^{-1}(z/cx)=\tfrac{1}{2}\ln(z+cx)-\tfrac{1}{2}\ln(cx-z)$$

we have finally

$$w(x,y,z)=A_\pm I_{\pm\nu}[l(x^2-z^2/c^2)^{\frac{1}{2}}]\sin\left[\tfrac{1}{2}i\nu\ln\left(\frac{cx-z}{cx+z}\right)\right]e^{ily}. \tag{37}$$

It remains to impose the boundary conditions. The upper boundary condition is clearly satisfied. To satisfy the lower condition requires knowledge of u. The equations of motion show
$$u_z=-(1/c^2)\,w_x. \tag{38}$$

The absence of an analytic integral for u appears at the time to preclude the exact application of the boundary condition. Let us note, however, that

$$u=O((1/c)\,w).$$

Then to $O(\gamma/c)$, we can replace the exact boundary condition (31) by

$$w=0 \quad \text{on} \quad z=-\gamma x \quad (x>0).$$

This requires that

$$I_{\pm\nu}[l(1-\gamma^2/c^2)^{\frac{1}{2}}x]\sin\left[\tfrac{1}{2}i\nu\ln\left(\frac{c+\gamma}{c-\gamma}\right)\right]=0,$$

satisfied if
$$\tfrac{1}{2}\nu=in\pi/\ln\Delta.$$

The vertical velocity is then

$$w(x,y,z)=A_\pm I_{\pm 2in\pi/\ln\Delta}\left[l\left(x^2-\frac{z^2}{c^2}\right)^{\frac{1}{2}}\right]\sin\left[\frac{n\pi}{\ln\Delta}\ln\left(\frac{cx-z}{cx+z}\right)\right]e^{ily}.$$

Because of the restriction $\gamma/c \ll 1$, $\ln\Delta$ is small, and the Bessel functions are of very high complex order.

Asymptotically, as $l(x^2-z^2/c^2)^{\frac{1}{2}} \to \infty$,

$$I_{\pm\nu}[l(x^2-z^2/c^2)^{\frac{1}{2}}] = \frac{\exp\{l(x^2-z^2/c^2)^{\frac{1}{2}}\}}{(2\pi)^{\frac{1}{2}}(x^2-z^2/c^2)^{\frac{1}{4}}}\left[1+O\left(\frac{1}{l(x^2-z^2/c^2)^{\frac{1}{2}}}\right)\right],$$

which are exponentially growing away from the apex. On the other hand

$$K_\nu[l(x^2-z^2/c^2)^{\frac{1}{2}}] = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\frac{\exp\{-l(x^2-z^2/c^2)^{\frac{1}{2}}\}}{(x^2-z^2/c^2)^{\frac{1}{4}}}\left[1+O\left(\frac{1}{l(x^2-z^2/c^2)^{\frac{1}{2}}}\right)\right]$$

which is decaying. This solution is an edge wave.

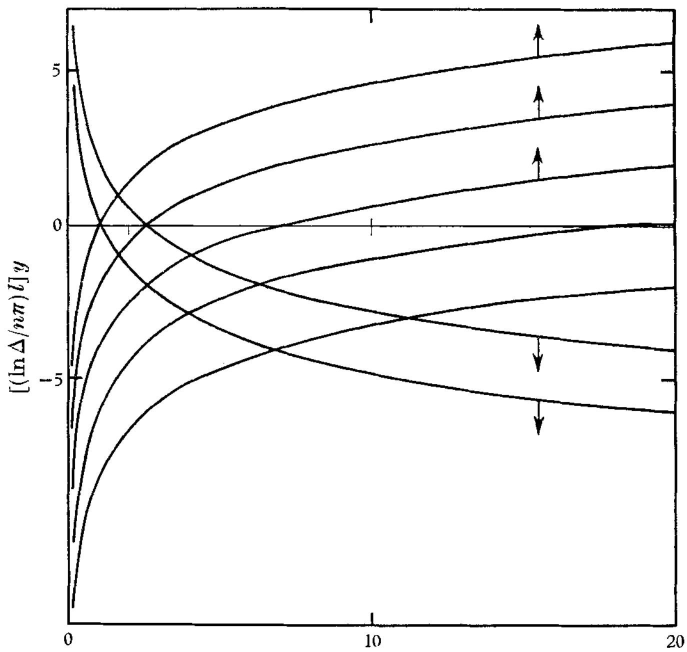

FIGURE 6. Line of constant phase along the plane $z/c = 0{\cdot}0$, $\gamma/c = 0{\cdot}1$. Valid for $l(x^2-z^2/c^2)^{\frac{1}{2}}$ small.

For small $l(x^2-z^2/c^2)^{\frac{1}{2}}$ we have asymptotically,

$$I_{\pm\nu}[l(x^2-z^2/c^2)^{\frac{1}{2}}] = l^{\pm\nu}\left(x^2-\frac{z^2}{c^2}\right)^{\pm\frac{1}{2}\nu}$$

$$w(x,y,z) \sim A_{\pm}\, l^{\pm 2in\pi/\ln\Delta}\exp\left\{\pm\frac{2in\pi}{\ln\Delta}\ln\left|x^2-\frac{z^2}{c^2}\right|\right\}\sin\left[\frac{n\pi}{\ln\Delta}\ln\left(\frac{cx-z}{cx+z}\right)\right]e^{ily}$$

and the lines of constant phase are given by

$$\pm\frac{n\pi}{\ln\Delta}\ln\left|x^2-\frac{z^2}{c^2}\right|+ly = \text{constant}.$$

The waves propagating toward the beach are obtained by choosing the $-\nu$ solutions.

Hence
$$\left[\frac{\ln\Delta}{n\pi}l\right]y = \ln\left|x^2-\frac{z^2}{c^2}\right|+\text{constant}. \tag{39}$$

For fixed z the phase lines are simply families of logarithmic curves. Two examples are shown in figures 6 and 7.

For $l[x^2 - z^2/c^2]^{\frac{1}{2}}$ large, the asymptotic form for I_ν given above shows that I_ν is real. Hence the phase lines are simply

$$ly = \text{constant}.$$

The waves are propagating with crests perpendicular to the beach. The y wave-number is fixed. As the waves approach the shallower water, the x wave-number

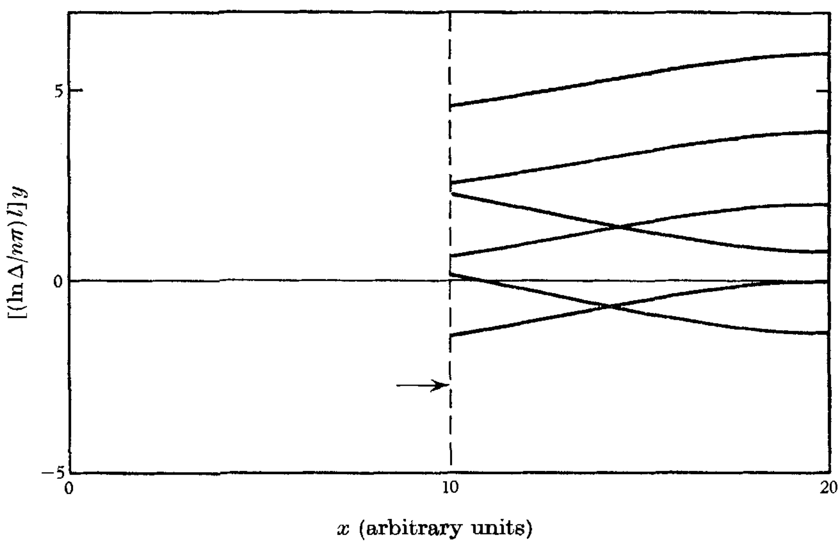

FIGURE 7. Lines of constant phase along the plane $z/c = 1{\cdot}0$, $\gamma/c = 0{\cdot}1$. Valid for $l(x^2 - x^2/c^2)^{\frac{1}{2}}$ small. Arrow denotes slope intersection.

starts to grow in such a way that the wave crests are steered more and more parallel to the slope. Along $z = 0$, the x wave-number reaches infinity in the corner. The refraction is, of course, very similar to that for surface waves, the waves in shallower water moving more slowly than those farther out, swinging the crests around parallel to the 'beach'.

The two-dimensional solutions treated in (I) and above, which were originally found by inspection, may be recovered from (34) by noting that in the case $l = 0$, the solutions are

$$\tilde{w}(r, \phi) = r^{\pm\nu} \sin \nu\phi.$$

6. Observations

There are few unambiguous observations of internal waves in the ocean. Most of those cases where clear-cut propagation directions have been observed (Lee 1961; Gaul 1961) propagation measured on the Continental Shelf has been toward the coast. This would be consistent with the picture presented here, which indicates that waves approaching the continental slope at an angle would be sharply refracted parallel to it. Of course, the waves could be generated at the edge of the slope; then one would expect them to have crests parallel to the continent.

The two-dimensional supercritical solutions predict a negative phase velocity both above and below the critical line, even though the energy flux reverses. No energy flux measurements appear to be available.

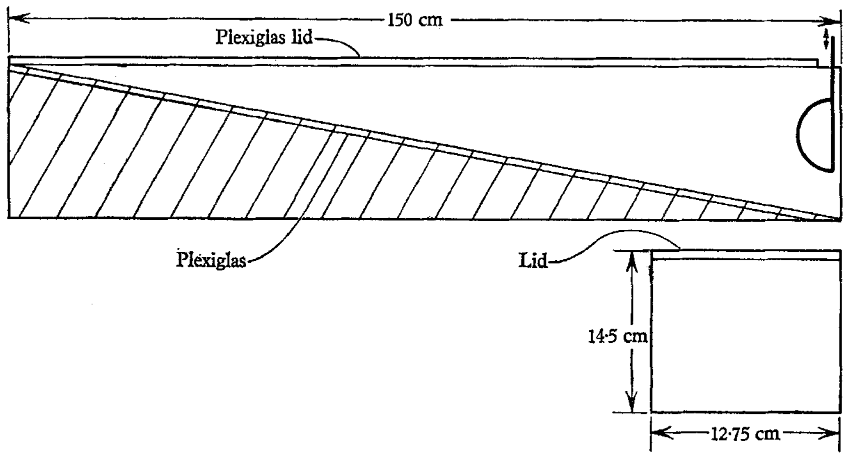

FIGURE 8. The plexiglass tank used for generating the modes on a wedge.

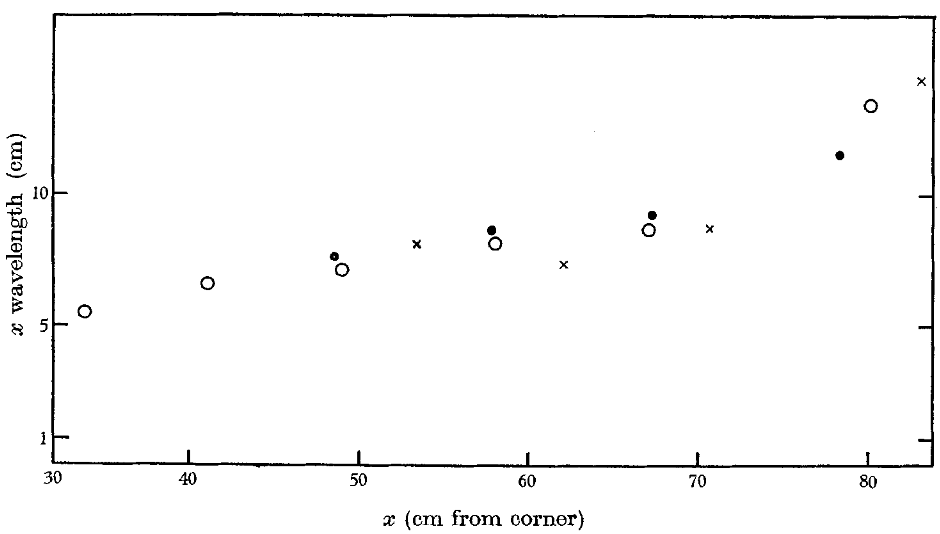

FIGURE 10. Apparent change in wavelength of the mode generated in figure 9, plate 1. A linear relationship appears reasonable far from the generation area. Different symbols refer to different instants of time. Wavelengths were arbitrarily referred to mid-points between crests. $c = 1{\cdot}68$, $\gamma = 0{\cdot}112$.

An attempt to test qualitatively some of the ideas given here in the laboratory has been made. A sketch of the apparatus is shown in figure 8. A salt-stratified fluid was placed in the wedge-shaped region, and a disturbance generated by the plunger at the deep end. The Brunt–Väisälä period was approximately 3 sec, and $\gamma/c = 0{\cdot}067$. A typical resulting wave-train is shown in figure 9, plate 1, with aluminum particles used for tracer. The wave-train appeared to be purely progressive, with the motion decaying up-slope.

Equation (11) indicates that locally the x wavelength dependence of the ψ^- modes should be

$$\lambda_x = \frac{2\pi}{q}\frac{(x^2 - z^2/c^2)}{x}.$$

For $x^2 \gg (z/c)^2$, this is a linear relation. In figure 10 is shown the apparent wavelength for the case depicted in figure 9, for $z/c \sim 2$. A roughly linear relation is obviously correct for small x. For x large, there is a deviation from linearity, but the motion is very complex in the vicinity of the wave-generator. Detailed confirmation of the theory must await more careful experiments.

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